JOURNAL OF APPROXIMATION THEORY 82, 340–351 (1995)

(T, E)-Korovkin Closures in Normed Spaces and BKW-Operators

SIN-EI TAKAHASI

Department of Basic Technology, Applied Mathematics and Physics, Yamagata University, Yonezawa 992, Japan

Communicated by Dany Leviatan

Received August 10, 1993; accepted in revised form September 7, 1994

DEDICATED TO PROFESSOR TAKAYUKI FURUTA ON HIS 60TH BIRTHDAY

We give a characterization of BKW-operators on special normed spaces and determine completely a class of BKW-operators from a function space on [0, 1] into $C(\Omega)$ for special test functions.

INTRODUCTION

In [8], the author introduced a class of operators on normed spaces satisfying a Bohman–Korovkin–Wulbert-type theorem and investigated such operators on special function spaces. We call such operators BKW. In this paper we investigate a certain Korovkin closure in normed spaces and give a characterization of BKW-operators from a normed space into a unital commutative C^*-algebra by means of the uniqueness sets (Theorem 1.3). Also applying this characterization, we completely determine a class of BKW-operators from a function space on the closed unit interval [0, 1] into the Banach space $C(\Omega)$ of all continuous complex-valued functions on a compact Hausdorff space Ω for the test functions $\{\mathbf{1}, x\}$ (Theorem 2.1), where a function space on [0, 1] means a subspace of $C([0, 1])$ which contains all constant functions and separates the points of [0, 1]. We also completely determine a class of norm one unital BKW-operators from a function space on [0, 1] into $C(\Omega)$ for the test functions $\{\mathbf{1}, x, x^2\}$ (Theorem 2.2).

1. (T, E)-KOROVKIN CLOSURES IN NORMED SPACES

Let X and Y be normed spaces and $B(X, Y)$ the normed space of all bounded linear operators from X into Y with the usual operator norm. Let

0021-9045/95 $12.00

E be a bounded subset of Y^*, the dual space of Y. For $S \subset X$ and $T \in B(X, Y)$, let $K_T^0(S; E)$ (resp. $K_T^1(S; E)$, $K_T^2(S; E)$) be the closed linear subspace of all $x \in X$ such that if $\{T_\lambda\}$ is a net of $B(X, Y)$ such that $\sup_\lambda \|T_\lambda\| \leqslant \|T\|$ (resp. $\lim_\lambda \|T_\lambda\| = \|T\|$, $\|T_\lambda\| = \|T\|$ $(\forall \lambda)$) and $\lim_\lambda \|T_\lambda(s) - T(s)\|_E = 0$ for all $s \in S$, then $\lim_\lambda \|T_\lambda(x) - T(x)\|_E = 0$. Here $\|y\|_E = \sup_{g \in E} |g(y)|$ for each $y \in Y$. We first show that all theses subspaces coincide.

LEMMA 1.1. *Under the above notations, we have* $K_T^0(S; E) = K_T^1(S; E) = K_T^2(S; E)$.

Proof. It is clear that $K_T^1(S; E) \subset K_T^2(S; E)$. To show $K_T^0(S; E) \subset K_T^1(S; E)$, let $x \in K_T^0(S; E)$ and let $\{T_\lambda\}$ be a net of $B(X, Y)$ such that $\lim_\lambda \|T_\lambda\| = \|T\|$ and $\lim_\lambda \|T_\lambda(s) - T(s)\|_E = 0$ for all $s \in S$. As we can assume $T \neq 0$, it follows that $T_\lambda \neq 0$ for sufficiently large λ. Put $S_\lambda = (\|T\|/\|T_\lambda\|) T_\lambda$. Then we can easily see that $\lim_\lambda \|T_\lambda(h) - S_\lambda(h)\|_E = 0$ for any $h \in X$. Hence $\lim_\lambda \|S_\lambda(s) - T(s)\|_E = 0$ for all $s \in S$, so $\lim_\lambda \|S_\lambda(x) - T(x)\|_E = 0$, since $\sup_\lambda \|S_\lambda\| = \|T\|$ and $x \in K_T^0(S; E)$. Therefore we obtain that $\lim_\lambda \|T_\lambda(x) - T(x)\|_E = 0$ and hence $x \in K_T^1(S; E)$.

We next show that $K_T^2(S; E) \subset K_T^0(S; E)$. Note that the lemma holds whenever $\overline{sp}(S) = X$ or $Y = \{0\}$ or $E = \{0\}$, so we can assume that $\overline{sp}(S) \neq X$, $Y \neq \{0\}$ and $E \neq \{0\}$, where $\overline{sp}(S)$ is the closed linear span of S in X. Then we can find a functional $f \in X^*$ and an element $y \in Y$ such that $\|f\| = 1$, $\|y\| = 1$, $f \mid \overline{sp}(S) = 0$ and $\|y\|_E \neq 0$. Put $(y \otimes f)(a) = f(a) y$ for each $a \in X$. Then $y \otimes f$ is a norm one linear operator from X into Y. Now let $x \in K_T^2(S; E)$ and consider a net $\{T_\lambda\}$ of $B(X, Y)$ such that $\sup_\lambda \|T_\lambda\| \leqslant \|T\|$ and $\lim_\lambda \|T_\lambda(s) - T(s)\|_E = 0$ for all $s \in S$. We need show that $\lim_\lambda \|T_\lambda(x) - T(x)\|_E = 0$. To do this let $\{T_{\lambda'}\}$ be any subnet of $\{T_\lambda\}$. Then $\lim_{\lambda'} \|T_{\lambda'}(s) - T(s)\|_E = 0$ for all $s \in S$. Fix λ' and for any real number $t \in \mathbf{R}$, put $\varphi(t) = \|T_{\lambda'} + t y \otimes f\|$. Then φ is continuous on $\mathbf{R}$ and

$$\varphi(0) = \|T_{\lambda'}\| \leqslant \|T\| \leqslant 2\,\|T\| - \|T_{\lambda'}\| \leqslant \|\pm 2\|T\| y \otimes f + T_{\lambda'}\| = \varphi(\pm 2\,\|T\|),$$

hence, by the intermediate value theorem, there exist $\alpha_{\lambda'} \in \mathbf{R}$ and $\beta_{\lambda'} \in \mathbf{R}$ such that

$$-2\,\|T\| \leqslant \beta_{\lambda'} \leqslant 0 \leqslant \alpha_{\lambda'} \leqslant 2\,\|T\| \qquad \text{and} \qquad \varphi(\beta_{\lambda'}) = \varphi(\alpha_{\lambda'}) = \|T\|.$$

Then there exist a convergent subnet $\{\alpha_{\lambda''}\}$ of $\{\alpha_{\lambda'}\}$ and a convergent subnet $\{\beta_{\lambda''}\}$ of $\{\beta_{\lambda'}\}$. For each λ'', set

$$U_{\lambda''} = T_{\lambda''} + \alpha_{\lambda''} y \otimes f \qquad \text{and} \qquad V_{\lambda''} = T_{\lambda''} + \beta_{\lambda''} y \otimes f.$$

Then $\|U_{\lambda''}\| = \|V_{\lambda''}\| = \|T\|$ and $U_{\lambda''}(s) = V_{\lambda''}(s) = T_{\lambda''}(s)$ for all $s \in S$. Since $x \in K_T^2(S; E)$, it follows that $\lim_{\lambda''} \|U_{\lambda''}(x) - T(x)\|_E = 0$ and

$\lim_{\lambda''} \|V_{\lambda''}(x) - T(x)\|_E = 0$. Hence we have that $\lim_{\lambda''} \|\alpha_{\lambda''} f(x)y - \beta_{\lambda''} f(x)y\|_E = 0$. Put $\alpha = \lim_{\lambda''} \alpha_{\lambda''}$ and $\beta = \lim_{\lambda''} \beta_{\lambda''}$. Then $\alpha \geqslant 0$, $\beta \leqslant 0$ and $\|(\alpha - \beta) f(x)y\|_E = 0$. If $f(x) = 0$, then the equality: $\lim_{\lambda''} \|U_{\lambda''}(x) - T(x)\|_E = 0$ means that $\lim_{\lambda''} \|T_{\lambda''}(x) - T(x)\|_E = 0$. If $f(x) \neq 0$, then $\alpha = \beta$, so $\alpha = \beta = 0$, hence the same equality: $\lim_{\lambda''} \|U_{\lambda''}(x) - T(x)\|_E = 0$ easily implies that $\lim_{\lambda''} \|T_{\lambda''}(x) - T(x)\|_E = 0$. We thus obtain that for any subnet $\{T_{\lambda'}\}$ of $\{T_\lambda\}$, there exists a subnet $\{T_{\lambda''}\}$ of $\{T_{\lambda'}\}$ such that $\lim_{\lambda''} \|T_{\lambda''}(x) - T(x)\|_E = 0$. In other words, $\lim_\lambda \|T_\lambda(x) - T(x)\|_E = 0$.

Q.E.D.

DEFINITION (cf. Altomare–Boccaccio [2] and Romanelli [7]). Let $K_T(S; E) = K_T^i(S; E)(i = 0, 1, 2)$ and we call $K_T(S; E)$ a (T, E)-Korovkin closure of S.

Remark 1. In case of that $X = Y = C(\Omega)$ (Ω is a compact Hausdorff space), $T =$ the identity operator and $E = \{\delta_\omega : \omega \in \Omega\}$, F. Altomare and C. Boccaccio have already proved that $K_T^0(S; E) = K_T^1(S; E) = K_T^2(S; E)$ (see [2, Theorem 1.2 and Remark 1.3]).

For $S \subset X$, $T \in B(X, Y)$ and $g \in E$, let $\hat{S}_{T,g}$ be the set of all $x \in X$ such that if $f \in X^*$, $\|f\| \leqslant \|T\|$ and $f(s) = g(T(s))$ for each $s \in S$, then $f(x) = g(T(x))$. Also set

$$\hat{S}_{T,E} = \bigcap_{g \in E} \hat{S}_{T,g}.$$

In this case, we have always that $\overline{sp}(S) \subset \hat{S}_{T,E}$.

Under these notations, we have the following result which contains [9, Lemma 2.1], applying the technique used by L. C. Kurtz [5], C. Micchelli [6], F. Altomare and C. Boccaccio [2], F. Altomare [1], etc.

THEOREM 1.1. *If E is a weak*-closed subset of the unit ball of Y^*, then* $\hat{S}_{T,E} \subset K_T(S; E)$.

Proof. Let x_0 be an arbitrary element of $\hat{S}_{T,E}$ and suppose that $\{T_\lambda : \lambda \in \Lambda\}$ is a net of $B(X, Y)$ such that $\sup_\lambda \|T_\lambda\| \leqslant \|T\|$ and $\lim_\lambda \|T_\lambda(s) - T(s)\|_E = 0$ for each $s \in S$. Then we must show that $\lim_\lambda \|T_\lambda(x_0) - T(x_0)\|_E = 0$. Assume the contrary. Then there is an $\varepsilon_0 > 0$ such that for any $\lambda \in \Lambda$, there exists $\alpha_\lambda \in \Lambda$ satisfying $\lambda \leqslant \alpha_\lambda$ and $\|T_{\alpha_\lambda}(x_0) - T(x_0)\|_E \geqslant \varepsilon_0$. Then we can find a functional $g_\lambda \in E$ such that

$$|g_\lambda(T_{\alpha_\lambda}(x_0)) - g_\lambda(T(x_0))| \geqslant \frac{\varepsilon_0}{2}. \qquad (*)$$

We can assume without loss of generality that the net $\{g_\lambda : \lambda \in \Lambda\}$ converges to a functional $g_0 \in E$ in the weak*-topology. For each $\lambda \in \Lambda$, set $f_\lambda(x) = g_\lambda(T_{\alpha_\lambda}(x))(x \in X)$. Then each f_λ is a functional in X^* such that $\|f_\lambda\| \leqslant \|T\|$. Also we can assume that the net $\{f_\lambda : \lambda \in \Lambda\}$ converges to a functional $f \in X^*$ with $\|f\| \leqslant \|T\|$ in the weak*-topology. For any $s \in S$,

$$|f_\lambda(s) - g_0(T(s))| \leqslant |g_\lambda(T_{\alpha_\lambda}(s)) - g_\lambda(T(s))| + |g_\lambda(T(s)) - g_0(T(s))|$$
$$\leqslant \|T_{\alpha_\lambda}(s) - T(s)\|_E + |g_\lambda(T(s)) - g_0(T(s))|.$$

Hence, after taking the limit with respect to λ, we obtain that $f(s) = g_0(T(s))$ for all $s \in S$. Therefore $f(x_0) = g_0(T(x_0))$ because $x_0 \in \hat{S}_{T,E} \subset \hat{S}_{T,g_0}$. However this contradicts the inequality $(*)$. Q.E.D.

Remark 2. Let $\bar{E}$ be the weak*-closure of $E \subset Y^*$. Then $K_T(S; \bar{E}) = K_T(S; E)$. However it seems that $\hat{S}_{T,\bar{E}} \neq \hat{S}_{T,E}$ in general because by the Hahn-Banach extension theorem, we have $\hat{S}_{T,\{0\}} = \overline{sp}(S)$ (if $T \neq 0$).

Applying the technique used by H. Bauer [3], C. Micchelli [6], F. Altomare and C. Boccaccio [2], F. Altomare [1] etc., we have the following result which characterizes (T, E)-Korovkin closures by means of the uniqueness sets in some special cases.

THEOREM 1.2. *Let X be a normed space, $S \subset X$, A a function algebra of continuous functions on a locally compact Hausdorff space Ω which contains the space of all continuous functions on Ω having compact support, $E \subset \Omega(\subset A^*)$ and $T \in B(X, A)$. Then $K_T(S; E) \subset \hat{S}_{T,E}$. In particular, if E is compact, then $\hat{S}_{T,E} = K_T(S; E)$.*

Proof. Let $x_0 \in K_T(S; E)$ and $\omega_0 \in E$. Suppose that f is a functional in X^* such that $\|f\| \leqslant \|T\|$ and $f(s) = (Ts)(\omega_0)$ for each $s \in S$. Let $\{U_\lambda : \lambda \in \Lambda\}$ be the set of all relatively compact open neighbourhoods of ω_0 in Ω. If $\lambda, \lambda' \in \Lambda$, we define $\lambda \leqslant \lambda'$ to mean that $U_{\lambda'} \subset U_\lambda$. Hence Λ is a direct set. For each $\lambda \in \Lambda$, choose an element $a_\lambda \in A$ such that $0 \leqslant a_\lambda(\omega) \leqslant 1$ for all $\omega \in \Omega$, $a_\lambda(\omega_0) = 1$ and $a_\lambda(\omega) = 0$ for all $\omega \in \Omega \backslash U_\lambda$, and set

$$T_\lambda(x) = f(x) a_\lambda + T(x)(1 - a_\lambda)$$

for each $x \in X$. Then each T_λ is a bounded linear operator from X into A such that $\|T_\lambda\| \leqslant \|T\|$. We claim that $\lim_\lambda \|T_\lambda(s) - T(s)\|_E = 0$ for all $s \in S$. Indeed, let $s \in S$. Then for any $\varepsilon > 0$, there exists $\lambda_\varepsilon \in \Lambda$ such that $|(Ts)(\omega_0) - (Ts)(\omega)| < \varepsilon$ for all $\omega \in U_{\lambda_\varepsilon}$. Hence, we have

$$|(T_\lambda s)(\omega) - (Ts)(\omega)| = a_\lambda(\omega)\, |f(s) - (Ts)(\omega)| < \varepsilon$$

for all $\omega \in \Omega$ and all $\lambda \geqslant \lambda_\varepsilon$. By taking the supremum over all $\omega \in \Omega$, we obtain that $\lim_\lambda \|T_\lambda(s) - T(s)\|_E = 0$. Therefore $\lim_\lambda \|T_\lambda(x_0) - T(x_0)\|_E = 0$ because $x_0 \in K_T(S; E)$. In particular, we have that $\lim_\lambda (T_\lambda x_0)(\omega_0) = (Tx_0)(\omega_0)$, hence $f(x_0) = (Tx_0)(\omega_0)$ since $(T_\lambda x_0)(\omega_0) = f(x_0)$ for all $\lambda \in \Lambda$. In other words, $x_0 \in \hat{S}_{T, \omega_0}$. Since ω_0 is an arbitrary point in E, it follows that $x_0 \in \hat{S}_{T, E}$. Consequently, $K_T(S; E) \subset \hat{S}_{T, E}$. If E is compact, then by Theorem 1.1, we obtain that $\hat{S}_{T, E} = K_T(S; E)$. Q.E.D.

For $S \subset X$ and $F \subset X^*$, let $U_S(F)$ be the set of all $g \in F$ such that if $f \in F$ and $f(s) = g(s)$ for all $s \in S$, then $f = g$. For $\rho > 0$, let $X_\rho^* = \{f \in X^* : \|f\| \leqslant \rho\}$. It is clear that the following three conditions are equivalent:

(1) $\hat{S}_{T, E} = X$.

(2) $\hat{S}_{T, g} = X$ for all $g \in E$.

(3) $T^*(E) \subset U_S(X^*_{\|T\|})$ (under the condition: $E \subset Y_1^*$).

For $S \subset X$, let $\mathrm{BKW}(X, Y; S, \|\ \|_E)$ be the set of all $T \in B(X, Y)$ such that if $\{T_\lambda\}$ is a net of $B(X, Y)$ satisfying $\lim_\lambda \|T_\lambda\| = \|T\|$ and $\lim_\lambda \|T_\lambda(s) - T(s)\|_E = 0$ $(\forall s \in S)$, then $\lim_\lambda \|T_\lambda(x) - T(x)\|_E = 0$ $(\forall x \in X)$. Therefore T belongs to $\mathrm{BKW}(X, Y; S, \|\ \|_E)$ if and only if $K_T(S; E) = X$.

Definition. We call an operator in $\mathrm{BKW}(X, Y; S, \|\ \|_E)$ a BKW-operator from X into Y for the test set S and the semi-norm $\|\ \|_E$. We will omit the semi-norm $\|\ \|_E$ when $\|y\| = \|y\|_E$ for all $y \in Y$ (cf. [8].)

The following result follows immediately from Theorem 1.2 and the above argument.

Theorem 1.3. *Let X be a normed space, $S \subset X$, A a function algebra of continuous functions on a locally compact Hausdorff space Ω which contains the space of all continuous functions on Ω having compact support, E a compact subset of $\Omega (\subset A^*)$, $T \in B(X, A)$ and T^* the adjoint of T. Then T is a BKW-operator for S and $\|\ \|_E$ if and only if $T^*(E) \subset U_S(X^*_{\|T\|})$.*

2. BKW-Operators from a Function Space on [0, 1] into $C(\Omega)$

Recall that an operator T from a function space into another function space is said to be unital if $T(\mathbf{1}) = \mathbf{1}$. By applying Theorem 1.3. we can completely determine all BKW-operators from a function space on $[0, 1]$ into $C(\Omega)$ for the test functions $\{\mathbf{1}, x\}$ $(x(t) = t$ for each $t \in [0, 1])$ as follows:

THEOREM 2.1. *Let Ω be a compact Hausdorff space and X a function space on $[0, 1]$ such that $sp\{\mathbf{1}, x\} \subsetneq X$, where* sp$\{\mathbf{1}, x\}$ *denotes the linear span of $\{\mathbf{1}, x\}$. Then every BKW-operator T from X into $C(\Omega)$ for the test functions $\{\mathbf{1}, x\}$ is of form*

$$T(f) = f(0)u + f(1)v$$

for every $f \in X$, where u and v are functions in $C(\Omega)$ satisfying the following two conditions:

(i) $|u(\omega)| + |v(\omega)| = \|T\|$ for all $\omega \in \Omega$.

(ii) If $u(\omega) \neq 0$ and $v(\omega) \neq 0$, then $|u(\omega) + v(\omega)| \neq \|T\|$.

In this case, the functions u and v are given by $u = T(\mathbf{1} - x)$ and $v = T(x)$.

In particular, every norm one unital BKW-operator T from X into $C(\Omega)$ for $\{\mathbf{1}, x\}$ is of form

$$T(f) = f(0)\chi + f(1)(\mathbf{1} - \chi)$$

for every $f \in X$, where χ is a characteristic function on some clopen subset of Ω.

Moreover, we can completely determine all norm one unital BKW-operators T from a function space on $[0, 1]$ into $C(\Omega)$ for the test functions $\{\mathbf{1}, x, x^2\}$ as follows:

THEOREM 2.2. *Let Ω be a compact Hausdorff space and X a function space on $[0, 1]$ such that $\{\mathbf{1}, x, x^2, x^3\} \subset X$. Then every norm one unital BKW-operator T from X into $C(\Omega)$ for the test functions $\{\mathbf{1}, x, x^2\}$ is of form*

$$(Tf)(\omega) = \begin{cases} f(\varphi(\omega)), & \text{if } \omega \in \Omega \backslash G \\ f(0)\{1 - \varphi(\omega)\} + f(1)\,\varphi(\omega), & \text{if } \omega \in G \end{cases}$$

for every $f \in X$, where φ is a continuous map from Ω into $[0, 1]$ and G is an open subset of Ω such that $0 < \varphi(\omega) < 1 (\forall \omega \in G)$ and $\varphi(\omega) = 0$ or $1 (\forall \omega \in \partial G)$. Here ∂G denotes the topological boundary of G in Ω. In this case, the function φ is given by $\varphi = T(x)$.

Remark 3. Let Φ and Ω be compact Hausdorff spaces and let X be a function space on Φ. If any bounded linear operator from X into $C(\Omega)$ has a norm preserving linear extension to the whole space $C(\Phi)$, then the restriction map: $T \to T \mid X$ maps BKW$(C(\Phi), C(\Omega); S)$ into BKW$(X, C(\Omega); S)$, where S denotes a set of test functions. Theorem 2.1 asserts that this restriction map is well-defined and onto in case of $\Phi = [0, 1]$ and $S = \{\mathbf{1}, x\}$. If we consider only the norm one unital operators, then Theorem 2.2 asserts that the restriction map is well-defined and onto in case of $\Phi = [0, 1]$ and $S = \{1, x, x^2\}$.

In order to prove the above theorems, we have to prepare some lemmas.

LEMMA 2.1. *Let X be a normed space and S a subset of X such that $\overline{sp}(S) \subsetneq X$. Then all functionals in $U_S(X_1^*)$ are of norm one.*

Proof. Let $f \in U_S(X_1^*)$. Assume that $\|f\| < 1$. By the Hahn–Banach separation theorem, we can find a nonzero functional $g \in X^*$ such that $g(s) = 0$ for all $s \in S$. Set $h = f + \lambda g$, where $\lambda = (1 - \|f\|)/\|g\|$. Then $h(s) = f(s)$ for all $s \in S$ and $\|h\| \leqslant 1$, hence $h = f$, a contradiction. Q.E.D.

The following result is well-known (cf. [4, Note 12.29]).

LEMMA 2.2. *Let (Ω, μ) be a measure space and $L^1(\Omega, \mu)$ the space of all complex-valued integrable functions on Ω. If $f_1, ..., f_n \in L^1(\Omega, \mu)$ are such that*

$$\left|\int_\Omega f_1(\omega)\, d\mu(\omega)\right| + \cdots + \left|\int_\Omega f_n(\omega)\, d\mu(\omega)\right|$$

$$= \int_\Omega |f_1(\omega)|\, d\mu(\omega) + \cdots + \int_\Omega |f_n(\omega)|\, d\mu(\omega),$$

then $f_1(\omega) = e^{i\theta_1} |f_1(\omega)|, ..., f_n(\omega) = e^{i\theta_n} |f_n(\omega)|$ μ-a.e. on Ω, where $\theta_j = \operatorname{Arg}[\int_\Omega f_j(\omega)\, d\mu(\omega)]$ $(j = 1, ..., n)$.

The following result completely determines the uniqueness set $U_{\{1, x\}}(X_1^*)$.

LEMMA 2.3. *Let X be a function space on $[0, 1]$ such that $sp\{1, x\} \subsetneq X$. Then $U_{\{1, x\}}(X_1^*) = \{a\,\delta_0 \mid X + b\,\delta_1 \mid X\colon a, b \in \mathbf{C}, |a| + |b| = 1$ and $|a + b| \neq 1$ (if $a \neq 0$, $b \neq 0$)$\}$.*

Proof. Let $\mu \in U_{\{1, x\}}(X_1^*)$. Put $a = \mu(1 - x)$ and $b = \mu(x)$. Then $|a| \leqslant 1$ and $|b| \leqslant 1$. For any $\alpha, \beta \in \mathbf{C}$, we have

$$\begin{aligned} |\alpha a + \beta b| &= |\mu\{\alpha(1 - x) + \beta x\}| \\ &\leqslant \|\mu\|\, \|\alpha(1 - x) + \beta x\| \\ &\leqslant \max_{0 \leqslant t \leqslant 1} |\alpha(1 - t) + \beta t|. \end{aligned}$$

In particular, for $\alpha = \bar{a}/|a|$ and $\beta = \bar{b}/|b|$, we have $|a| + |b| \leqslant \max_{0 \leqslant t \leqslant 1}\{|\alpha|\,(1 - t) + |\beta|\, t\} = 1$. Now set $\nu = a\,\delta_0 \mid X + b\,\delta_1 \mid X$ and so

$\|\nu\| \leqslant |a| + |b| \leqslant 1$. Also $\nu(\mathbf{1}) = a + b = \mu(\mathbf{1})$ and $\nu(x) = b = \mu(x)$. Then $\mu = \nu$, i.e., $\mu = a\,\delta_0 | X + b\,\delta_1 | X$ because $\mu \in U_{\{\mathbf{1}, x\}}(X_1^*)$. Moreover by Lemma 2.1, $\|\mu\| = 1$, so that $1 \leqslant |a| + |b|$ and hence we have $|a| + |b| = 1$. If also $a \neq 0$ and $b \neq 0$, then $|a + b| \neq 1$. In fact, if $|a + b| = 1$, then we can find $t > 0$ such that $b = ta$. Also choose a function $g \in X \backslash sp\{\mathbf{1}, x\}$ and put $f = g - g(0)\mathbf{1} + \{g(0) - g(1)\}\,x$. Then $f \in X$ and $f \neq 0$, hence there exists $0 < s < 1$ such that $f(s) \neq 0$. Note that $(s - t + st)/s < 1$, so take a positive number ρ such that $\max\{0, (s - t + st)/s\} < \rho < 1$. Set

$$\alpha = \rho a,\ \beta = \frac{(1-\rho)a}{1-s},\ \gamma = \frac{(1-s)b - s(1-\rho)a}{1-s}$$

and

$$\mu_1 = \alpha\,\delta_0 \mid X + \beta\,\delta_s \mid X + \gamma\,\delta_1 \mid X.$$

Then we can easily see that $\mu_1(\mathbf{1}) = \mu(\mathbf{1})$ and $\mu_1(x) = \mu(x)$. Also since $(s - t + st)/s < \rho$,

$$\begin{aligned}
|\alpha| + |\beta| + |\gamma| &= \rho\,|a| + \frac{(1-\rho)\,|a|}{1-s} + \frac{|(1-s)\,t - s(1-\rho)|\,|a|}{1-s} \\
&= |a| \left\{ \rho + \frac{1-\rho}{1-s} + \frac{(1-s)\,t - s(1-\rho)}{1-s} \right\} \\
&= |a|\,(1+t) = |a| + |b| = 1,
\end{aligned}$$

hence $\|\mu_1\| \leqslant 1$. Then $\mu_1 = \mu$ because $\mu \in U_{\{\mathbf{1}, x\}}(X_1^*)$. However $\mu_1(f) = \beta f(s) \neq 0$ and $\mu(f) = af(0) + bf(1) = 0$, so $\mu_1 \neq \mu$. This is a contradiction.

Conversely, let $a, b \in \mathbf{C}$ be such that $|a| + |b| = 1$ and $|a + b| \neq 1$ if $a \neq 0$, $b \neq 0$. We must show that $a\,\delta_0 | X + b\,\delta_1 | X \in U_{\{\mathbf{1}, x\}}(X_1^*)$. To do this, let $\mu \in X_1^*$ be such that $\mu(\mathbf{1}) = a + b$ and $\mu(x) = b$. By the Hahn–Banach extension theorem, we can find a Radon measure $\tilde{\mu}$ on $[0, 1]$ such that $\tilde{\mu} \mid X = \mu$ and $\|\tilde{\mu}\| = \|\mu\|$. Let $\tilde{\mu} = u\,|\tilde{\mu}|$ be the polar decomposition of $\tilde{\mu}$, i.e.,

$$\int_0^1 f(t)\,d\tilde{\mu}(t) = \int_0^1 f(t)\,u(t)\,d\,|\tilde{\mu}|\,(t)$$

for all $f \in L^1([0, 1], |\tilde{\mu}|)$, where $|\tilde{\mu}|$ is the total variation of $\tilde{\mu}$ and u is a measurable function on $[0, 1]$ with $|u(t)| = 1$ for all $t \in [0, 1]$ (see [4, Corollary 19.38]). Then we have the following inequality:

$$\begin{aligned}1 = |a| + |b| &= |\mu(1-x)| + |\mu(x)| \\ &= \left|\int_0^1 (1-t)\, u(t) d\,|\tilde{\mu}|\,(t)\right| + \left|\int_0^1 tu(t) d\,|\tilde{\mu}|\,(t)\right| \\ &\leqslant \int_0^1 (1-t) d\,|\tilde{\mu}|\,(t) + \int_0^1 td\,|\tilde{\mu}|\,(t) \\ &\leqslant \int_0^1 d\,|\tilde{\mu}| = \|\tilde{\mu}\| = \|\mu\| \leqslant 1.\end{aligned}$$

If $a \neq 0$ and $b \neq 0$, then by Lemma 2.2, we have $(1-t)\, u(t) = e^{i\alpha}(1-t)(|\tilde{\mu}| - \text{a.e.})$ and $tu(t) = e^{i\beta}t(|\tilde{\mu}| - \text{a.e.})$, where $\alpha = \text{Arg}(a)$ and $\beta = \text{Arg}(b)$. Hence we have

$$1 = |(1-t)\, e^{i\alpha} + te^{i\beta}|\ (|\tilde{\mu}| - a.e.).$$

Since $|a+b| \neq 1$ and hence $\alpha \neq \beta\ (\text{mod.}\ 2\pi)$, it follows that $|\tilde{\mu}|([0,1] \backslash \{0,1\}) = 0$, i.e., $\text{supp}(|\tilde{\mu}|) \subset \{0, 1\}$ by the above equation. If $a = 0$, then the above inequality implies that $\int_0^1 (1-t)\, d\,|\tilde{\mu}|\,(t) = 0$ and hence $\text{supp}(|\tilde{\mu}|) = \{1\}$. If $b = 0$, the same inequality implies that $\int_0^1 td\,|\tilde{\mu}|\,(t) = 0$ and hence $\text{supp}(|\tilde{\mu}|) = \{0\}$. Then $|\tilde{\mu}|$ can be expressed as $|\tilde{\mu}| = c\,\delta_0 + d\,\delta_1$, for some complex numbers c and d. Therefore $\tilde{\mu} = cu(0)\,\delta_0 + du(1)\delta_1$, hence we can easily see that $\mu = a\,\delta_0 \mid X + b\,\delta_1 \mid X$. We thus obtain that $a\,\delta_0 \mid X + b\,\delta_1 \mid X \in U_{\{1,x\}}(X_1^*)$. Q.E.D.

The following result completely determines the "positive" functionals in the uniqueness set $U_{\{1,x,x^2\}}(X_1^*)$. The same result has been obtained by C. Micchelli for $X = C([0,1])$ (see [6, Theorem 3.1]).

LEMMA 2.4. *Let X be a function space on $[0,1]$ such that $\{1, x, x^2, x^3\} \subset X$ and $X_+^* = \{\mu \in X^*\colon \|\mu\| = \mu(1)\}$. Then $U_{\{1,x,x^2\}}(X_1^*) \cap X_+^* = \{\delta_\lambda \mid X\colon 0 \leqslant \lambda \leqslant 1\} \cup (1-a)\,\delta_0 \mid X + a\,\delta_1 \mid X\colon 0 < a < 1\}$.*

Proof. Let $0 \leqslant a \leqslant 1$. Then $\delta_a \mid X$ and $(1-a)\,\delta_0 \mid X + a\,\delta_1 \mid X$ are in X_+^*. To show that $\delta_a \mid X \in U_{\{1,x,x^2\}}(X_1^*)$, let $v \in X_1^*$ be such that $v(x^k) = \delta_a(x^k)(k = 0, 1, 2)$. Then $1 = v(1) \leqslant \|v\| \leqslant 1$. Choose a Radom measure $\tilde{v}$ on $[0,1]$ such that $\tilde{v} \mid X = v$ and $\|\tilde{v}\| = \|v\|$. Then $\|\tilde{v}\| = \tilde{v}(1) = 1$, so $\tilde{v}$ is positive and also we have

$$\tilde{v}((x-a)^2) = v(x^2) - 2av(x) + a^2v(1) = a^2 - 2a^2 + a^2 = 0.$$

Hence, the support of $\tilde{v}$ is the single point $\{a\}$. This immediately implies that $\tilde{v} = \delta_a$, so $v = \delta_a \mid X$ and hence $\delta_a \mid X \in U_{\{1,x,x^2\}}(X_1^*)$. Next, to show that $(1-a)\,\delta_0 \mid X + a\,\delta_1 \mid X \in U_{\{1,x,x^2\}}(X_1^*)$, let $v \in X_1^*$ be such that $v(x^k) = ((1-a)\,\delta_0 + a\,\delta_1)(x^k)(k = 0, 1, 2)$. Then $1 = v(1) \leqslant \|v\| \leqslant 1$.

Choose a Radom measure $\tilde{\nu}$ on $[0, 1]$ such that $\tilde{\nu} \mid X = \nu$ and $\|\tilde{\nu}\| = \|\nu\|$. Then $\tilde{\nu}$ is positive and $\tilde{\nu}(x - x^2) = \nu(x - x^2) = a - a = 0$. Hence, the support of $\tilde{\nu}$ is contained in $\{0, 1\}$. This immediately implies that $\nu = (1-a)\,\delta_0 \mid X + a\,\delta_1 \mid X$ and hence $(1-a)\,\delta_0 \mid X + a\,\delta_1 \mid X \in U_{\{1, x, x^2\}}(X_1^*)$.

Conversely, let $\mu \in U_{\{1, x, x^2\}}(X_1^*) \cap X_+^*$. By Lemma 2.1, $\|\mu\| = 1$, and so $\mu(\mathbf{1}) = 1$. Choose a positive Radon measure $\tilde{\mu}$ on $[0, 1]$ such that $\tilde{\mu} \mid X = \mu$ and $\|\tilde{\mu}\| = \|\mu\|$. Put $\alpha = \mu(x)$ and $\beta = \mu(x^2)$. Then we have that $0 \leqslant \alpha, \beta \leqslant 1$, $\beta \leqslant \alpha$ and $\alpha^2 \leqslant \beta$ by Schwarz's inequality. If $\alpha = \beta$, then $\tilde{\mu}(x - x^2) = 0$ and hence we have $\mu = (1-\alpha)\,\delta_0 \mid X + \alpha\,\delta_1 \mid X$ by the above argument. If also $\alpha^2 = \beta$, then $\tilde{\mu}((x-\alpha)^2) = \beta - 2\alpha^2 + \alpha^2 = 0$ and so $\mu = \delta_\alpha \mid X$ by the above argument. We finally show that the case $0 < \alpha^2 < \beta < \alpha < 1$ does not occur. Assume the contrary. For each $0 < \lambda < 1$, set

$$\mu_\lambda = a(\lambda)\,\delta_0 \mid X + b(\lambda)\,\delta_\lambda \mid X + c(\lambda)\,\delta_1 \mid X,$$

where $a(\lambda) = \lambda^{-1}\{\lambda - (1+\lambda)\,\alpha + \beta\}$, $b(\lambda) = (\lambda(1-\lambda))^{-1}(\alpha - \beta)$ and $c(\lambda) = (1-\lambda)^{-1}(\beta - \lambda\alpha)$. Then we have that $\mu_\lambda(\mathbf{1}) = 1 = \mu(\mathbf{1})$, $\mu_\lambda(x) = \alpha = \mu(x)$ and $\mu_\lambda(x^2) = \beta = \mu(x^2)$. Note that $a(\alpha) > 0$, $b(\alpha) > 0$ and $c(\alpha) > 0$. Then $\|\mu_\alpha\| = 1$ and hence μ_α must equal μ. Now choose a number ε such that $0 < \varepsilon < \min\{1 - \alpha, (\beta - \alpha^2)/\alpha\}$. Then $a(\alpha + \varepsilon) > 0$, $b(\alpha + \varepsilon) > 0$ and $c(\alpha + \varepsilon) > 0$. Hence $\|\mu_{\alpha+\varepsilon}\| = 1$, so $\mu_{\alpha+\varepsilon}$ must equal μ. On the other hand, we can find a polynomial $p \in X$ of the third degree such that $p(0) = p(1) = p(\alpha) = 0$ and $p(\alpha + \varepsilon) \neq 0$. Then we have $\mu_{\alpha+\varepsilon}(p) = b(\alpha+\varepsilon)\,p(\alpha+\varepsilon) \neq 0$ and $\mu_\alpha(p) = b(\alpha)\,p(\alpha) = 0$. This is a contradiction. Q.E.D.

Remark 4. In Lemma 2.4 and Theorem 2.2, we can replace the condition: $\{\mathbf{1}, x, x^2, x^3\} \subset X$ by the condition: X contains a Chebyshev system $\{\mathbf{1}, x, x^2, g\}$ of order 3.

The following result is fundamental and its proof is straightforward, and so is left to the reader.

LEMMA 2.5. *Let Ω be a topological space, G an open subset of Ω. Let φ and ψ be continuous maps from Ω to another topological space such that $\varphi(\omega) = \psi(\omega)$ for each $\omega \in \partial G$ and let f be defined on Ω by*

$$f(\omega) = \begin{cases} \psi(\omega) & \text{if } \omega \in \Omega \backslash G \\ \varphi(\omega), & \text{if } \omega \in G. \end{cases}$$

Then f is continuous on Ω. Here ∂G denotes the topological boundary of G.

Proof of Theorem 2.1. Let T be a bounded linear operator from X into $C(\Omega)$. Without loss of generality, we can assume that T is of norm one. By Theorem 1.3, T is a BKW-operator from X into $C(\Omega)$ for the test functions $\{\mathbf{1}, x\}$ if and only if $T^*(\delta_\omega) \in U_{\{1, x\}}(X_1^*)$ for all $\omega \in \Omega$, where δ_ω denotes the evaluation at $\omega \in \Omega$. Also by Lemma 2.3, $T^*(\delta_\omega) \in U_{\{1,x\}}(X_1^*)$ for all $\omega \in \Omega$ if and only if for each $\omega \in \Omega$, there exists a pair of complex numbers $(u(\omega), v(\omega))$ such that $T^*(\delta_\omega) = u(\omega)\, \delta_0 \mid X + v(\omega)\, \delta_1 \mid X$, $|u(\omega)| + |v(\omega)| = 1$ and $|u(\omega) + v(\omega)| \neq 1$ when $u(\omega) \neq 0$ and $v(\omega) \neq 0$. Note that $T^*(\delta_\omega) = u(\omega)\, \delta_0 \mid X + v(\omega)\, \delta_1 \mid X$ means that $(Tf)(\omega) = f(0)u(\omega) + f(1)v(\omega)$ for all $f \in X$. We thus obtain that $T(f) = f(0)u + f(1)v$ for all $f \in X$. Moreover, this equation easily implies that $u = T(\mathbf{1} - x)$ and $v = T(x)$ and so u and v are in $C(\Omega)$.

In particular, if T is unital, we have

$$1 = (T\mathbf{1})(\omega) = u(\omega) + v(\omega)$$

for all $\omega \in \Omega$. Therefore $\Omega = \Omega_u \cup \Omega_v$ and $\Omega_u \cap \Omega_v = \varnothing$, where $\Omega_u = \{\omega \in \Phi\colon u(\omega) \neq 0\}$ and $\Omega_v = \{\omega \in \Phi\colon v(\omega) \neq 0\}$. Hence u and v equal the characteristic functions on Ω_u and Ω_v, respectively. Of course, $u + v = \mathbf{1}$, so that by putting $\chi = u$, we obtain that the desired equation:

$$T(f) = f(0)\chi + f(1)(\mathbf{1} - \chi)$$

for every $f \in X$. Q.E.D.

Proof of Theorem 2.2. Let T be a norm one unital BKW-operator from X into $C(\Omega)$ for the test functions $\{\mathbf{1}, x, x^2\}$. Let $\omega \in \Omega$. Then by Theorem 1.3, $T^*(\delta_\omega) \in U_{\{\mathbf{1},x,x^2\}}(X_1^*)$ and so $\|T^*(\delta_\omega)\| = 1$ by Lemma 2.1. Note also that $(T^*\delta_\omega)(1) = \mathbf{1}(\omega) = 1$. Therefore $T^*(\delta_\omega) \in U_{\{\mathbf{1},x,x^2\}}(X_1^*) \cap X_+^*$ for all $\omega \in \Omega$. Hence by Lemma 2.4, we have $\Omega = F_T \cup G_T$, where F_T is the set of all $\omega \in \Omega$ such that $T^*(\delta_\omega) \in \{\delta_\lambda \mid X\colon 0 \leqslant \lambda \leqslant 1\}$ and G_T is the set of all $\omega \in \Omega$ such that $T^*(\delta_\omega) \in \{(1-a)\, \delta_0 \mid X + a\, \delta_1 \mid X\colon 0 < a < 1\}$. Since the map: $\omega \to T^*(\delta_\omega)$ is weak*-continuous and the set $\{\delta_\lambda \mid X\colon 0 \leqslant \lambda \leqslant 1\}$ is weak*-closed, F_T must be closed. Of course, $F_T \cap G_T = \varnothing$, hence G_T is open. For each $\omega \in F_T$, we can find a unique point $t(\omega)$ in $[0, 1]$ such that $T^*(\delta_\omega) = \delta_{t(\omega)} \mid X$. Also for each $\omega \in G_T$, we can find a unique point $s(\omega)$ in the open unit interval $]0, 1[$ such that $T^*(\delta_\omega) = (1 - s(\omega))\, \delta_0 \mid X + s(\omega)\, \delta_1 \mid X$. Note that $(Tx)(\omega) = t(\omega)$ for each $\omega \in F_T$ and $(Tx)(\omega) = s(\omega)$ for each $\omega \in G_T$. Then for each $f \in X$, we have

$$(Tf)(\omega) = \begin{cases} f((Tx)(\omega)), & \text{if } \omega \in F_T \\ f(0)\{1 - (Tx)(\omega)\} + f(1)(Tx)(\omega), & \text{if } \omega \in G_T. \end{cases}$$

Let $\omega \in \partial G_T$. Then there exists a net $\{\omega_\lambda\}$ in G_T which converges to ω. Set $t = (Tx)(\omega)$ and $t_\lambda = (Tx)(\omega_\lambda)$. Then $T^*(\delta_\omega) = \delta_t \mid X$, $T^*(\delta_{\omega_\lambda}) = (1 - t_\lambda)\, \delta_0 \mid X + t_\lambda\, \delta_1 \mid X$ and the net $\{t_\lambda\}$ converges to t. Since $w^* - \lim_\lambda T^*(\delta_{\omega_\lambda}) = T^*(\delta_\omega)$, it follows that $\delta_t \mid X = (1-t)\, \delta_0 \mid X + t\, \delta_1 \mid X$, hence, $\delta_t(x^2) = (1-t)\, \delta_0(x^2) + t\, \delta_1(x^2)$, so $t = 0$ or 1.

Conversely, let φ be a continuous map from Ω into $[0, 1]$ and G is an open subset of Ω such that $0 < \varphi(\omega) < 1 (\forall \omega \in G)$ and $\varphi(\omega) = 0$ or $1 (\forall \omega \in \partial G)$. For each $f \in X$, put

$$(T_\varphi f)(\omega) = \begin{cases} f(\varphi(\omega)), & \text{if } \omega \in \Omega \backslash G \\ f(0)\{1 - \varphi(\omega)\} + f(1)\, \varphi(\omega), & \text{if } \omega \in G. \end{cases}$$

Since $\varphi(\omega) = 0$ or $1 (\forall \omega \in \partial G)$, it follows that $f(0)\{1 - \varphi(\omega)\} + f(1)\, \varphi(\omega) = f(\varphi(\omega))$ for all $\omega \in \partial G$. Then for each $f \in X$, $T_\varphi(f)$ is a complex-valued continuous function on Ω by Lemma 2.5. Moreover we can easily see that T_φ is a norm one unital linear operator from X into $C(\Omega)$. Also by the definition of T_φ, we have that

$$T_\varphi^*(\delta_\omega) \in \{\delta_\lambda \mid X: 0 \leqslant \lambda \leqslant 1\} \bigcup \{(1-a)\, \delta_0 \mid X + a\, \delta_1 \mid X: 0 < a < 1\}$$

for all $\omega \in \Omega$. Then T_φ is BKW for the test functions $\{\mathbf{1}, x, x^2\}$ from Theorem 1.3 and Lemma 2.4. Q.E.D.

ACKNOWLEDGMENTS

The author thanks one of the referees for a helpful suggestion on Theorem 1.2, and the other referees for valuable comments. The author also expresses his appreciation to Dr. Keiji Minagawa for useful advice.

REFERENCES

1. F. ALTOMARE, Approximation of finitely defined operators in function spaces, *Note Mat.* **7** (1987), 211–229.
2. F. ALTOMARE AND C. BOCCACCIO, On Korovkin-type theorems in spaces of continuous complex-valued functions, *Boll. Un. Mat. Ital. B (6)* **1**, No. 1 (1982), 75–86.
3. H. BAUER, Theorems of Korovkin type for adapted spaces, *Ann. Inst. Fourier (Grenoble)* **23** (1973), 245–260.
4. E. HEWITT AND K. STROMBERG, "Real and Abstract Analysis," Springer-Verlag, Berlin/Heidelberg/New York, 1969.
5. L. C. KURTZ, Unique Hahn–Banach extension and Korovkin's theorem, *Proc. Amer. Math. Soc.* **47** (1975), 413–416.
6. C. A. MICCHELLI, Convergence of positive linear operators on $C(X)$, *J. Approx. Theory* **13** (1975), 305–315.
7. S. ROMANELLI, Universal Korovkin closures with respect to operators on commutative Banach algebras, *Math. Japon.* **37**, No. 3 (1992), 427–443.
8. S.-E. TAKAHASI, Bohman–Korovkin–Wulbert operators on normed spaces, *J. Approx. Theory* **72** (1993), 174–184.
9. S.-E. TAKAHASI, BKW-operators on function spaces, *Rend. Circ. Mat. Palermo (2) Suppl.* **33** (1993), 479–488.